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3 ANALOGUE TECHNIQUE IN SPECTRAL ANALYSIS

3.1 SPECTRAL ANALYSIS BASED ON ANALOG TECHNIQUE

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INTRODUCTION

Data available in an analog form as fluctuating voltages can be spectrally analysed using either digital or analog technique.

With an analog technique the analysis is done directly on the fluctuating voltages, whereas before using a digital technique, we have to digitise with unavoidable loss of information.

The normal method for analog spectral analysis is based on the use of bandpass filtering, on which we shall concentrate in this paper.

GENERAL THEORY

We shall first introduce the necessary concepts and definitions in the way most suitable for our purpose. Let $x(t)$ be the signal we want to analyse. We assume that $x(t)$ is a stationary, random function of the time, t, for $-\infty < t < \infty$. We assume moreover that $x(t)$ has a zero mean value

$$\overline{x(t)} = 0$$

where the overbar denotes ensemble average.

$x(t)$ can be Fourier-expanded using stochastic Fourier-Stieltje integrals [1, 2]

$$x(t) = \int_{-\infty}^{\infty} e^{i\omega t} \, dZ_x(\omega)$$
$$Z_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-i\omega t}}{it} \, x(t) \, dt \qquad (1)$$

The reason for this representation is that $x(t)$, being a stationary, random function, does not fulfil the conditions for the normal Fourier expansion to be valid. $Z_x(\omega)$ is a stochastic function, which also averages to zero. Moreover, nonoverlapping increments of $Z_x(\omega)$ are orthogonal, while overlapping increments provide the power spectral differential. The latter statements may be written

$$\iint_{-\infty}^{\infty} \overline{dZ_x(\omega_1) \, dZ_x^*(\omega)} = \iint_{-\infty}^{\infty} S_{xx}(\omega) \, \delta(\omega - \omega_1) \, d\omega \, d\omega_1 \qquad (2)$$

where $\delta(\omega)$ is the Dirac delta-function and the asterisk denotes complex conjugation. $S_{xx}(\omega)$ is the power spectrum.

Let $y(t)$ be another stationary, random function with zero mean. Equivalent to equation (1), we have

$$y(t) = \int_{-\infty}^{\infty} e^{i\omega t} \, dZ_y(\omega)$$
$$Z_y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-i\omega t}}{it} \, y(t) \, dt \qquad (3)$$

$Z_y(\omega)$ fulfils the same conditions as $Z_x(\omega)$. Moreover, $Z_x(\omega)$ and $Z_y(\omega)$ fulfil a condition corresponding to equation (2).

$$\iint_{-\infty}^{\infty} \overline{dZ_x^*(\omega_1) \, dZ_y(\omega)} = \iint_{-\infty}^{\infty} S_{xy}(\omega) \, \delta(\omega - \omega_1) \, d\omega_1 \, d\omega \qquad (4)$$

$$= \int_{-\infty}^{\infty} \left(Co_{xy}(\omega) + i Q_{xy}(\omega) \right) d\omega$$

where we have written the correlation spectrum, $S_{xy}(\omega)$, in terms of the co-spectrum, $Co_{xy}(\omega)$ and the quadrature spectrum, $Q_{xy}(\omega)$.

Fig 3.1.1. Linear system.

Finally is defined what we shall understand by a filter. The filters we are concerned with are linear systems. We consider in Fig 3.1.1 a linear system with an input signal, $x(t)$, an output signal, $x_1(t)$, and a unit impulse response function, $k(t)$. The passage of a signal through such a system can be expressed by a convolution integral

$$x_1(t) = \int_{-\infty}^{\infty} k(\tau)\, x(t-\tau)\, d\tau \qquad (5)$$

Equation (5) is a general equation for a linear system. It is valid for both a digital and an analog filter, i.e. an electronic circuit. An analog filter must however be physically realizable, meaning that an input signal $x(t)$ can give no contribution to the output signal until it has arrived at the input terminal. Consequently, we see from equation (5) that for an analog filter we must have

$$k(\tau) = 0 \quad \text{for} \quad \tau < 0 \qquad (6)$$

In spite of this we will keep the lower limit of the integral in equation (5), partly for convenience but also because our results then will be valid for digital filtering too. We now turn to the principal filter set-up for spectral analysis, which is shown in Fig 3.1.2.

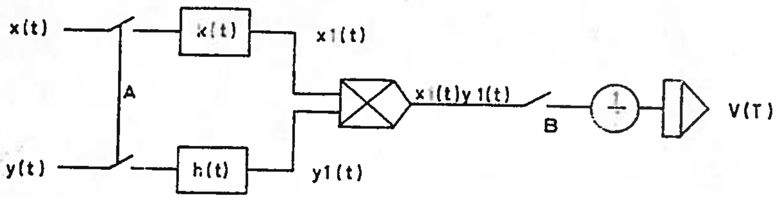

Fig 3.1.2. The principle for spectral analysis

$$V(T) = \frac{1}{T}\int_{-T/2}^{+T/2} x_1(t)\, y_1(t)\,dt$$

In Fig 3.1.2 the two signals, $x(t)$ and $y(t)$, are to be analysed. They are assumed to exist as stationary, random functions for $-\infty < t < \infty$. Switch A is used to take out a finite record of length T. The two filters are in general different, having unit impulse response functions $k(t)$ and $h(t)$. The two filter outputs, $x_1(t)$ and $y_1(t)$, are multiplied together and the product is time-averaged over the record length, T.

Switch B is used to start and stop the integration. In the model we will use here, B is operated simultaneously with A. Other models are of course possible [3, 5], for example that the integration starts so long before and goes on for so long after the record is passed through the filters that we can assume that the integration is performed for $-\infty < t < \infty$.

To treat the operation of the switches analytically, we introduce a data window function, $b(t)$.

$$b(t) = \begin{cases} 1 & \text{for } |t| \leq \frac{T}{2} \\ 0 & \text{for } |t| > \frac{T}{2} \end{cases} \tag{7}$$

From equations (5) and (7) we see that the filter outputs can be written

$$\begin{aligned} x_1(t) &= \int_{-\infty}^{\infty} k(\tau)\, b(t-\tau)\, x(t-\tau)\, d\tau \\ y_1(t) &= \int_{-\infty}^{\infty} h(\tau)\, b(t-\tau)\, y(t-\tau)\, d\tau \end{aligned} \tag{8}$$

Next we introduce the Fourier transforms of $b(t)$, $k(t)$ and $h(t)$.

$$\begin{aligned} b(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t}\, B(\omega)\, d\omega \\ B(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t}\, b(t)\, dt = \frac{\sin(\frac{\omega}{2} T)}{\frac{\omega}{2}} \end{aligned} \tag{9}$$

$$\begin{aligned} k(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t}\, K(\omega)\, d\omega \\ K(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t}\, k(t)\, dt \end{aligned} \tag{10}$$

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} H(\omega)\, d\omega \tag{11}$$

$$H(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} h(t)\, dt$$

$B(\omega)$ is shown in Fig 3.1.3. It is seen that $B(\omega)$ becomes sharper as T becomes bigger. As a matter of fact it can be shown that

$$\lim_{T\to\infty} B(\omega) = 2\pi\, \delta(\omega) \tag{12}$$

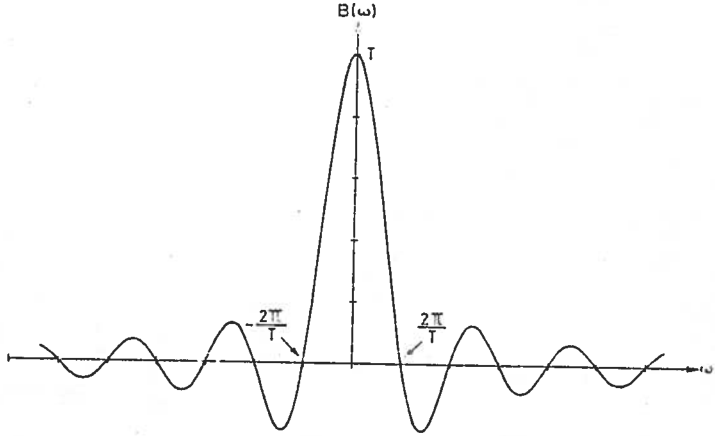

Fig 3.1.3. $B(\omega) = \sin \frac{\omega}{2}T / \frac{\omega}{2}$

$K(\omega)$ and $H(\omega)$ are called filter transfer functions. For analog filters they always have a real as well as an imaginary part. For bandpass filters the absolute values, $|K(\omega)|$ and $|H(\omega)|$, have high values in a bandpass around the frequencies $\omega = \pm\, \omega_0$ and values close to zero for all other frequencies. ω_0 is called the centre frequency.

As bandpass filters one may for example use filters with transfer functions like the one shown in Fig 3.1.4.

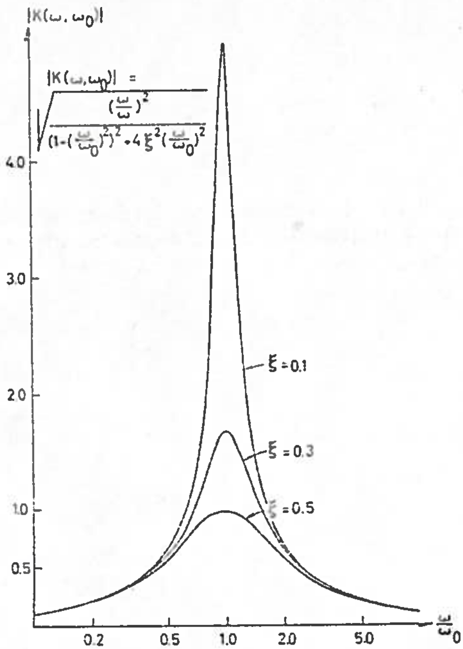

Fig 3.1.4. The absolute value of a bandpass transfer function $K(\omega, \omega_0)$, ξ is equal to $\frac{1}{2}\frac{\Delta\omega}{\omega_0}$, where $\Delta\omega$ is the width at the half-power point. ξ is called the relative bandwidth.

Combining equations (1) and (3) with equations (8) to (11) we obtain

$$
\begin{aligned}
x_1(t) &= \frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{i\omega_1 t} K(\omega_1)\, B(\omega_1 - \omega)\, dZ_x(\omega)\, d\omega_1 \\
y_1(t) &= \frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{i\omega_1 t} H(\omega_1)\, B(\omega_1 - \omega)\, dZ_y(\omega)\, d\omega_1
\end{aligned}
\qquad (13)
$$

Since $x_1(t)$ is a real quantity, we may write V(T), from Fig 3.1.2, as

$$V(T) = \frac{1}{T} \int_{-T/2}^{T/2} x_1^*(t)\, y_1(t)\, dt \qquad (14)$$

Introducing equation (13) into equation (14) we obtain after some rearranging

$$V(T) = \left(\frac{1}{2\pi}\right)^2 \frac{1}{T} \iiiint_{-\infty}^{\infty} B(\omega_2 - \omega_1)\, B(\omega_1 - \omega_3)\, B(\omega_2 - \omega_4) \cdot$$
$$K^*(\omega_1)\, H(\omega_2)\, dZ_x^*(\omega_3)\, dZ_y(\omega_4)\, d\omega_1\, d\omega_2 \qquad (15)$$

To simplify this we assume ergodicity and furthermore assume that T is so big that V(T) is close to $\overline{V(T)}$. If equation (15) is averaged we get, using equation (4)

$$\overline{V(T)} = \left(\frac{1}{2\pi}\right)^2 \frac{1}{T} \iiint_{-\infty}^{\infty} B(\omega_2 - \omega_1)\, B(\omega_1 - \omega)\, B(\omega_2 - \omega) \cdot$$
$$K^*(\omega_1)\, H(\omega_2)\, S_{xy}(\omega)\, d\omega_1\, d\omega_2\, d\omega \qquad (16)$$

Equation (16) can formally be simplified by introducing

$$HT(\omega) = \left(\frac{1}{2\pi}\right)^2 \frac{1}{T} \iint_{-\infty}^{\infty} B(\omega_2 - \omega_1)\, B(\omega_1 - \omega)\, B(\omega_2 - \omega)\, K^*(\omega_1)\, H(\omega_2) d\omega_1\, d\omega_2 \qquad (17)$$

Equation (16) now takes the form

$$\overline{V(T)} = \int_{-\infty}^{\infty} HT(\omega)\, S_{xy}(\omega)\, d\omega \qquad (18)$$

Equations (17) and (18) are the basic equations for spectral analysis by means of bandpass filters.

To simplify further we note that

$$HT(\omega) \to K^*(\omega)\, H(\omega) \quad \text{as } T \to \infty$$

If we moreover for a given finite T limit ourselves to filters for which $K^*(\omega)H(\omega)$ varies very little over frequency intervals of the order of $2\pi/T$, the width of $B(\omega)$, we see that ω_1 and ω_2 can be integrated out of equation (17), so that we are again left with $HT(\omega) = K^*(\omega)H(\omega)$.

That $K^*(\omega)H(\omega)$ varies very little over frequency intervals of the order of $2\pi/T$ we will express by saying that the width of $K^*(\omega)H(\omega)$ is much bigger than $2\pi/T$. The width we may think upon as the width at the half power point. It is also possible to carry out the following calculations without this limitation, but not much is gained by making the width of $K^*(\omega)\,H(\omega)$ small compared with the width of $B(\omega)$, as can be seen from equation (17).

As a consequence of the discussion above, we write equation (18) as

$$\overline{V(T)} = \int_{-\infty}^{\infty} K^*(\omega)\,H(\omega)\,S_{xy}(\omega)\,d\omega \qquad (19)$$

where equation (19) is valid only if the width of $K^*(\omega)\,H(\omega)$ is much bigger than $2\pi/T$.

Based on equation (19) we shall now in more detail study how the different types of spectra can be estimated.

CO-SPECTRA

We choose the filters so that $K^*(\omega)\,H(\omega) = T(\omega, \omega_o)$, where $T(\omega, \omega_o)$ is a real, positive and even function of ω, with peaks at $\omega = \pm\,\omega_o$ and close to zero elsewhere. This can for example be achieved by letting $H(\omega) = K(\omega) = K(\omega, \omega_o)$, where $|K(\omega, \omega_o)|$ is the function shown in Fig 3.1.4.

Substituting this into equation (19), we get

$$\overline{V(T)} = \int_{-\infty}^{\infty} T(\omega, \omega_o)\,S_{xy}(\omega)\,d\omega$$

$$= \int_{-\infty}^{\infty} T(\omega, \omega_o)\,(\,Co_{xy}(\omega) + i\,Q_{xy}(\omega))\,d\omega$$

$$= \int_{0}^{\infty} 2\,T(\omega, \omega_o)\,Co_{xy}(\omega)\,d\omega \qquad (20)$$

The last equality evolves because $Q_{xy}(\omega)$ is an odd function, and $C_{xy}(\omega)$ is an even function. We therefore only need to determine the spectra for $\omega \geq 0$.

In equation (20) the last integral gets most of its contribution from a bandpass around ω_o. If we assume that this bandpass is so narrow that $Co_{xy}(\omega)$ only varies insignificantly across it, $Co_{xy}(\omega)$ can be taken outside the integration sign.

$$\overline{V(T)} = Co_{xy}(\omega_o) \int_0^{\infty} 2T(\omega, \omega_o)\, d\omega \tag{21}$$

Finally, since we assume that $\overline{V(T)}$ is close to $V(T)$, we can estimate $Co_{xy}(\omega_o)$ from

$$Co_{xy}(\omega_o) \simeq V(T) \Big/ \int_0^{\infty} 2T(\omega, \omega_o)\, d\omega \tag{22}$$

POWER SPECTRA

We will use the same filter set-up as for estimation of co-spectra, $T(\omega,\omega_o) = |K(\omega,\omega_o)|^2$.

We now simply feed the same signal, $x(t)$, into the two filters in Fig 3. 1. 2.

The power spectrum at frequency ω_o can then be found from the equations for co-spectral estimation, replacing $Co_{xy}(\omega_o)$ by $S_{xx}(\omega_o)$.

$$\overline{V(T)} = \int_0^{\infty} 2T(\omega, \omega_o)\, S_{xx}(\omega)\, d\omega \simeq S_{xx}(\omega_o) \int_0^{\infty} 2T(\omega,\omega_o)\, d\omega \tag{23}$$

$$S_{xx}(\omega_o) \simeq V(T) \Big/ \int_0^{\infty} 2T(\omega,\omega_o)\, d\omega \tag{24}$$

A practical simplification of the power spectral analysing can be seen from Fig 3.1.2. Since we here have $y_1(t) = x_1(t)$ it is possible to use only one filter and then square the output before integrating.

QUADRATURE SPECTRA

We choose $K^*(\omega)H(\omega) = -iT_1(\omega,\omega_o)$, where $T_1(\omega,\omega_o)$ is a real, odd function with peaks at $\omega = \pm\omega_o$ and close to zero elsewhere. $T_1(\omega,\omega_o) \geq 0$ for $\omega \geq 0$. Substituting into equation (19) we obtain

$$V(T) = \int_{-\infty}^{\infty} -i\,T\,(\omega, \omega_o)\,(\, Co_{xy}(\omega) + i\,Q_{xy}(\omega)\,)\, d\omega \tag{25}$$

With arguments similar to those leading to equations (21) and (22) we get

$$\overline{V(T)} = \int_0^\infty 2T_1(\omega,\omega_o)\, Q_{xy}(\omega)\, d\omega \simeq Q_{xy}(\omega_o) \int_0^\infty 2T_1(\omega,\omega_o)\, d\omega \qquad (26)$$

$$Q_{xy}(\omega_o) \simeq V(T) \Big/ \int_0^\infty 2T(\omega,\omega_o)\, d\omega \qquad (27)$$

STATISTICAL CONFIDENCE

In deriving the expressions (22), (24) and (27), we have assumed that $V(T) \simeq \overline{V(T)}$.

We are however only allowed to assume that $V(T) = \overline{V(T)}$ in the limit $T \to \infty$, and then only if we assume stationarity and ergodicity

For finite T one may therefore ask how big a deviation of $V(T)$ from $\overline{V(T)}$ one may expect.

Since $V(T)$ is a random variable, it is convenient to express this deviation in terms of a variance.

$$\sigma_T^2 = \overline{(\, V(T) - \overline{V(T)}\,)^2} \qquad (28)$$

If we prefer to calculate the variance on the spectral estimate rather than on $V(T)$, we have

$$\sigma^2_{T\ \mathrm{spectrum}} = \sigma_T \Big/ \left(\int_0^\infty 2 |\, K^*(\omega) H(\omega)\, |\, d\omega \right)^2$$

Substituting equations (15) and (16) into equation (28) we get

$$\sigma_T^2 = \left(\left(\frac{1}{2\pi}\right)^2 \frac{1}{T}\right)^2 \overline{\left[\iiiint_{-\infty}^{\infty} B(\omega_2 - \omega_1)\, K^*(\omega_1)\, H(\omega_2)\, B(\omega_1 - \omega_2) \cdot \right.} \qquad (29)$$

$$\overline{\left.(B(\omega_2 - \overline{\omega_4})\, dZ_x^*(\omega_3)\, dZ_y(\omega_4) - B(\omega_2 - \omega_3)\, S_{xy}(\omega_3)\, \delta(\omega_3 - \omega_4)\, d\omega_3\, d\omega_4) d\omega_1 d\omega_2 \right]^2}$$

This squared four-dimensional integral is converted to an eight-dimensional integral, thus introducing a fourth order moment

$dZ_x^*(\omega_3)\, dZ_y(\omega_4)\, dZ_x^*(\omega_5)\, dZ_y(\omega_6)$

To reduce this, it is customary to assume that the signals $x(t_1)$, $x(t_2)$, $y(t_3)$ and $y(t_4)$ are joint Gaussian variables. This means that fourth order moments for both the time signals and the corresponding Fourier increments can be expressed by second order moments.

We moreover make the same assumptions which we made in order to arrive at equations (22), (24) and (27), namely, that the width of $K^*(\omega)\,H(\omega)$ is much bigger than $2\pi/T$, and that the spectra only vary negligibly across the bandpass of $K^*(\omega)\,H(\omega)$.

In the reduction of equation (29) it is convenient to introduce the so-called noise bandwidth, W, given by

$$W = \left(\int_0^\infty |K^*(\omega)\,H(\omega)|\; d\omega \right)^2 \Big/ \int_0^\infty |K^*(\omega)\,H(\omega)|^2\; d\omega \qquad (30)$$

W is for linear bandpass filters close to the other commonly used definitions of bandwidth, for example the half power bandwidth.

Introducing the definitions and assumptions made above into equation (29), we can after some calculations write down the formulas for the statistical variance of the different spectral estimates [5].

Power spectra

$$\sigma^2_{TS_{xx}(\omega_o)} = \frac{2\pi}{TW}\,(S_{xx}(\omega_o))^2 \qquad (31)$$

Co-spectra

$$\sigma^2_{TCo_{xy}(\omega_o)} = \frac{\pi}{TW}\,(S_{xx}(\omega_o)\,S_{yy}(\omega_o) + Co^2_{xy}(\omega_o) - Q^2_{xy}(\omega_o)) \qquad (32)$$

Quadrature spectra

$$\sigma^2_{TQ_{xy}(\omega_o)} = \frac{\pi}{TW}\,(S_{xx}(\omega_o)\,S_{yy}(\omega_o) + Q^2_{xy}(\omega_o) - Co^2_{xy}(\omega_o)) \qquad (33)$$

From equations (31), (32) and (33), we see that the statistical confidence in general is worse for cross-spectra than for power spectra.

If a more detailed estimation of the statistical confidence of a spectral estimate is wanted, one has to introduce additional assumptions.

For power spectral estimates it is often argued that they are distributed as a multiple of a chi-square variate [3, 5] with k degrees of freedom. k can be determined from

$$k = 2 \frac{\text{mean value } (\chi_k^2)}{\text{variance } (\chi_k^2)} = 2T \frac{W}{2\pi} \qquad (34)$$

The last equality is obtained by substituting $S_{xx}(\omega_o)$ for the mean value and equation (31) for the variance. The number of degrees of freedom for a power-spectral estimate is therefore equal to two times the record length times the noise bandwidth, measured in cycles/second, and it is independent of the spectral value itself.

With k given, any confidence interval for the estimate can be found from standard tables.

Although the assumptions used to arrive at equations (31) to (34) seem highly restrictive, the formulas work quite well in practice, when the spectra are smooth and $W >> 2\pi/T$.

If the spectra have peaks in the frequency intervals of interest, other formulas have been derived [3].

If W is not much bigger than $2\pi/T$, more complicated expressions must be used [5].

AVERAGING ERRORS

The averaging errors are the errors which are introduced when the spectrum is taken outside the integration sign as the spectral value at the centre frequency of the filter, as for example is done in equation (23). Still with reference to equation (23) this is strictly speaking only allowed if $T(\omega, \omega_o)$ is proportional to $\delta(\omega - \omega_o)$, or alternatively if $S_{xx}(\omega_o)$ is independent of ω_o.

The averaging errors are therefore dependent on the shape of both the filter transfer function and the spectrum. If the shape of the spectrum is approximately known, the averaging errors can be estimated for a given combination of filters.

Assume that we for example have a power spectrum of an analytical form, $S(\omega)$.

The estimate of $S(\omega)$ at frequency ω_o, $S_b(\omega_o)$, which is a result of a filtering procedure may be found using equation (23)

$$S_b(\omega_o) = \int_0^\infty T(\omega,\omega_o)\, S(\omega)\, d\omega \Big/ \int_0^\infty T(\omega,\omega_o)\, d\omega \tag{35}$$

Averaging errors are shown in Fig 3.1.5 for different spectra. The filter type used in the illustration is the one shown in Fig 3.1.4, i.e. $T(\omega, \omega_o) = |K(\omega,\omega_o)|^2$. The meaning

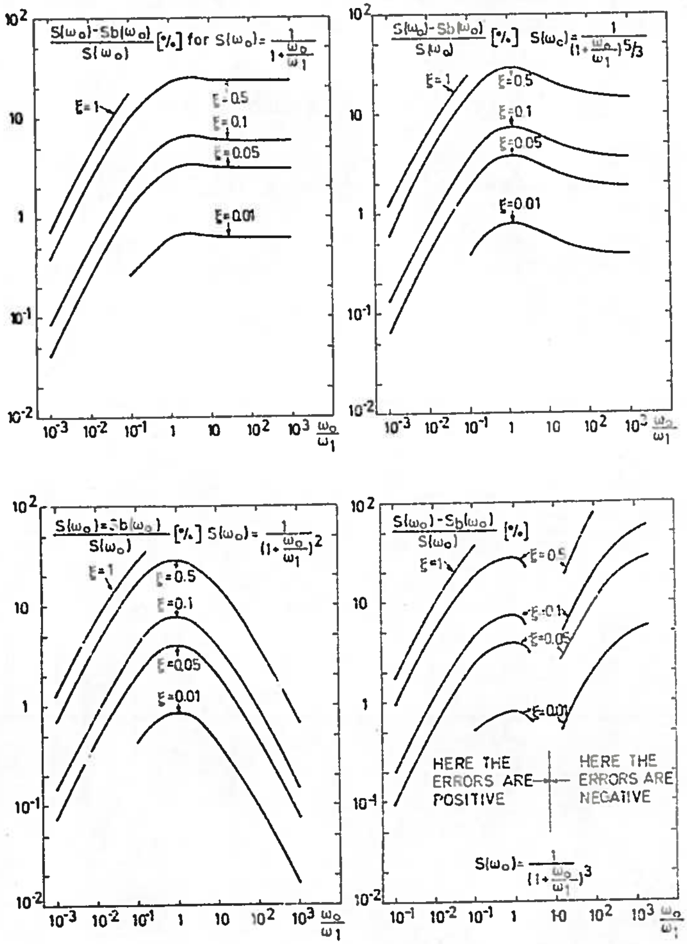

Fig 3.1.5. The averaging errors for different spectral forms. The filters are of the type shown in Fig 3.1.4.

of ξ is also shown in Fig 3.1.4.

It should be noted that if the spectrum is a smooth function in a logarithmic plot, then the averaging errors are essentially dependent on the relative filter width for a given spectral shape, rather than on the absolute filter width.

Attention is also drawn to the fact that the errors calculated in Fig 3.1.5 are for smooth spectra. If the spectra contain peaks, it is obvious that the width of the transfer functions must be smaller than the width of the spectral peaks, if the details of the spectrum shall not be completely lost.

CHOICE OF FILTERS

Fig 3.1.6 shows a composite power spectrum for the longitudinal velocity component of a turbulent field. The filter transfer functions used to estimate it are also shown. The spectrum is plotted as a function of n (cycles/second) rather than of ω (radians/second). Since the spectrum is plotted as

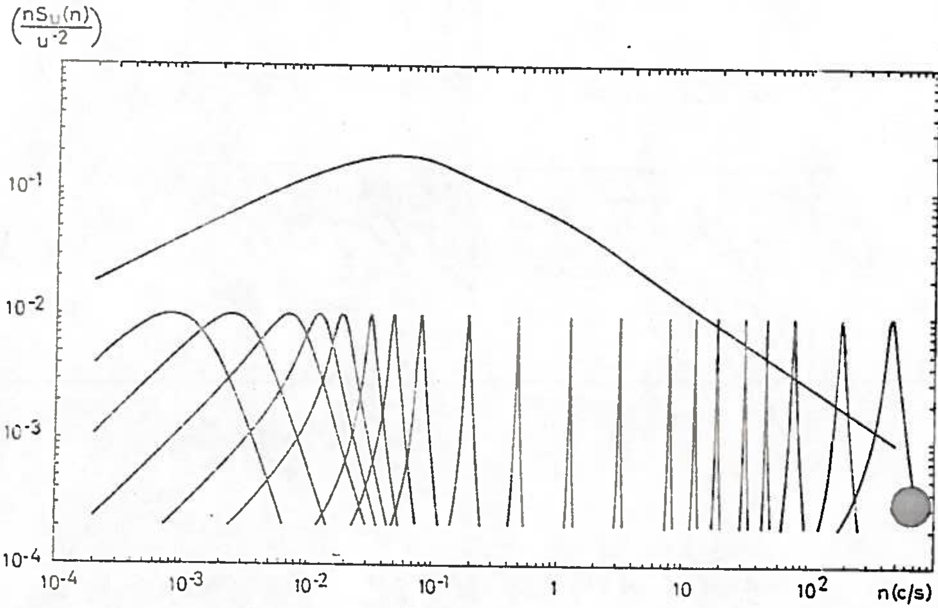

Fig 3.1.6. Power spectrum, with the transfer functions of the filters used to estimate the spectrum.

$nS(n)$, it is also the transfer functions for $nS(n)$ which are plotted. Equation (35 may be written

$$n_o S_b(n_o) = \int_0^\infty \frac{T(n, n_o)}{n} \, nS(n) \, dn \Big/ \int_0^\infty \frac{T(n, n_o)}{n_o} \, dn$$

What is plotted in Fig 3.1.6 is $\dfrac{T(n, n_o)}{n}$, properly normalized.

If we assume that we have determined the filter type we are going to use, we still have to determine the optimal filter width for the filters. To do so we must weigh a number of more or less contradictory requirements against each other. The most important of these are:

1) The filter width should be much bigger than $2\pi/T$, or $1/T$, if we are working with cycles/second.

2) From equations (31), (32) and (33) we see that for a given record length we shall make the transfer functions as broad as possible in order to achieve high statistical confidence.

3) The averaging errors should be as small as possible. From Fig 3.1.5 we see that we have to make the filter width small.

Points 1) to 3) have been thoroughly discussed above; we can moreover add the following points:

4) It is important that the spectral estimates at different frequencies are statistically independent of each other. This means that the overlap between neighbouring transfer functions has to be as small as possible.

5) On the other hand, we also want as many spectral estimates as possible to make the spectrum well determined. Therefore we will be tempted to place the transfer functions as close as possible.

Fig 3.1.6 is an example of how one may compromise between these five requirements.

The record length for the signals used to produce Fig 3.1.6 is one hour. This means that the filter transfer function at the lowest frequency is close to $1/T$ (cycles/second). Therefore point 1) has the highest priority at the low frequency end. As the frequency increases, we can increase the noise bandwidth and decrease the relative bandwidth of the transfer functions simultaneously, thus satisfying points 1) to 4) better and better.

In the region $0.1 < n < 10$ we were looking for the beginning of a power law. Therefore the independency of the estimates became very important. The transfer functions therefore are widely spaced.

That the transfer functions are broadening again, in the high frequency end, is due to electronic problems [5], which do not concern this discussion.

PREWHITENING

The weak link in analog treatment of data is that most analog recorders have a relatively poor signal to noise ratio.

Since spectra decrease in the high frequency region, one risks that this region will drown in the recorder noise. This can be avoided by amplitication of the high frequency part of the spectra.

From a recorder point of view, the best spectrum is a white noise spectrum, since the dynamic range of the recorder then can be fully utilized at all frequencies of the signal.

One therefore often tries to approximate the signal spectrum to a white noise spectrum before recording. Hence the name: prewhitening.

Prewhitening is however also an advantage from the filtering point of view, since a white noise spectrum has no averaging errors.

An often used prewhitening circuit is a differentiator, for which the output spectrum is equal to the input spectrum multiplied by the frequency squared.

The reason for using a differentiator is that it is a simple circuit and that many spectra of interest fall off in the high frequency region as a power law with an exponent close to minus two.

THE PRACTICAL PROGRAMMING

One can follow two ways for the practical set up of filter programs: Construct the filters oneself or buy one of the frequency analysers commercially available.

In [4] and [5] is described how filter programs can be constructed using an analog computer. The same technique can also be used without such a computer, which in these cases mostly is used as a convenient supply of electronic circuits.

Many frequency analysers use a heterodyning technique, bringing the frequency band to be studied to a fixed filter, rather than tuning the filter across a wide frequency band.

If we multiply $x(t)$ and $y(t)$ by $e^{i\Delta\omega t}$ and filter the signals, using identical filters, the filter outputs will be

$$x_1(t) = \int_{-\infty}^{\infty} k(\tau)\, x(t-\tau)\; e^{i\Delta\omega(t-\tau)}\, d\tau$$
$$y_1(t) = \int_{-\infty}^{\infty} k(\tau)\, y(t-\tau)\; e^{i\Delta\omega(t-\tau)}\, d\tau \qquad (36)$$

where we have used equation (8) and neglected the effects of the finite record length. If we introduce the relevant Fourier transforms into equation (36), we get

$$x_1(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, K(\omega)\, dZ_x(\omega + \Delta\omega)$$
$$y_1(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, K(\omega)\, dZ_y(\omega + \Delta\omega) \qquad (37)$$

Thus

$$\overline{V(T)} = \overline{x_1^*(t)\, y_1(t)} = \int_{-\infty}^{\infty} |\, K(\omega, \omega_o)\,|^2\, S_{xy}(\omega + \Delta\omega)\, d\omega \qquad (38)$$

ω_o is introduced to stress that $K(\omega)$ is a bandpass filter. Notice that we in this case also can use the special bandpass filter with $\omega_o=0$, i.e. a lowpass filter.

When the original time series are multiplied by a sine function and a cosine function for some fixed frequency, $\Delta\omega$, and each product is passed through a filter, the outputs are the real and imaginary parts of equation (36).

The sum of the squares of the real and the imaginary parts for each signal form, when time averaged, the power spectral estimate. The cross product of the real parts summed with that of the imaginary parts gives, after time averaging, the co-spectral estimate. The cross product of the real part for one signal summed with the other like product provides, after time averaging, the quadrature spectral estimate.

With the heterodyning technique we can therefore not only do with one fixed filter for the whole frequency range, we can also use the same filter type to provide estimates of all three types of spectra, while we in the more direct filter technique described in equations (20) to (27) need different filters to estimate the quadrature spectrum and the co- and power spectrum.

We pay for these advantages by the distortions introduced by the nonideality of the sine - cosine generator. Note moreover that if we want to use different filter widths for different frequencies, we must also in heterodyning change the filters. A description of the practical build up of a heterodyne filter system on an analog computer can be found in [4].

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